Q). use the method of Lagrangian multipliers to solve the following NLPP.

$$\min.\ z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$$

s.t.

$$x_1 + x_2 + x_3 = 20$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

Here, $f(X) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

$h(X) = x_1 + x_2 + x_3 - 20 = 0$

Let λ be a constant called Lagrangian multiplier, then the Lagrangian fun^n is

$$L(X,\lambda) = f(X) - \lambda h(X)$$

$$= (2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100) - \lambda(x_1 + x_2 + x_3 - 20)$$

$\therefore$ The necessary cond^n for $f(X)$ or z to be maximum or minimum are

$$\frac{\partial L}{\partial x_1} = 4x_1 + 10 - \lambda = 0 \Rightarrow x_1 = \frac{\lambda - 10}{4} \quad \text{--- (1)}$$

$$\frac{\partial L}{\partial x_2} = 2x_2 + 8 - \lambda = 0 \Rightarrow x_2 = \frac{\lambda - 8}{2} \quad \text{--- (2)}$$

$$\frac{\partial L}{\partial x_3} = 6x_3 + 6 - \lambda = 0 \Rightarrow x_3 = \frac{\lambda - 6}{6} \quad \text{--- (3)}$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 20 = 0 \quad \text{--- (4)}$$

putting the values of x_1, x_2, x_3 from eq^ns 1, 2 and 3 in eq^n 4, we get

$$\frac{\lambda-10}{4} + \frac{\lambda-8}{2} + \frac{\lambda-6}{6} = 20$$

$$\Rightarrow \lambda = 30$$

$$\therefore x_1 = \frac{30-10}{4} = 5$$

$$x_2 = \frac{30-8}{2} = 11$$

$$x_3 = \frac{30-6}{6} = 4$$

$\therefore$ the stationary pt. is $(5, 11, 4, 30)$

Now, the Bordered Hessian Matrix

$$H^B = \left[\begin{array}{c|ccc} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \hline \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{array}\right]$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{bmatrix}$$

Here $n = 3 \rightarrow$ no. of variables.
$m = 1 \rightarrow$ no. of constraints.

$2m+1 = 3$
$n-m = 3-1 = 2$.

Now, the principal minor of order $(2m+1)$ i.e. 3, the last $n-m = 2$ principal minors of H_0^B are

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -1(2) + 1(0-4) = -2-4 = -6 < 0$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = -44 < 0$$

and $(-1)^m = (-1)^1 = -1 < 0$.

Since the signs of Δ_3 and Δ_3 are of the sign of $(-1)^m$

Hence, $f(x)$ is min. at $(5, 11, 4)$ & min $f(x) = 281$ //

Another way to solve

Max^m or Min^m of $f(X)$ may also be checked as follows:

Hessian matrix of Objective function $f(X)$ at the point $(5, 11, 4)$ is

$$H(X) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Principal minor of $H(X)$ of order one $= |4| = 4$

of order 2 $= \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8$

and of order 3 $= \begin{vmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 48$